

On the power graphs which are Cayley graphs of some groups

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Abstract

In 2013, Jemal Abawajy, Andrei Kelarev and Morshed Chowdhury [1] proposed a problem to characterize the finite groups whose power graphs are Cayley graphs of some groups. Here we give a complete answer to this question.

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1 Introduction

Let G be a finite group. The concept of directed power graph $\vec{P}(G)$ was introduced by Kelarev and Quinn [4]. $\vec{P}(G)$ is a digraph with the vertex set G and for $x, y \in G$, there is an arc from x to y if and only if $x \neq y$ and $y = x^m$ for some positive integer m . Then Chakraborty, Ghosh and Sen [2] defined undirected power graph $\mathcal{P}(G)$ of a group G , which is a simple graph having the same vertex set as $\vec{P}(G)$ and two distinct vertices in G are adjacent if and only if either of them is a positive power of the other. The undirected power graph $\mathcal{P}(G)$ of a finite group G is complete if and only if G is a cyclic p -group [Theorem 2.12, [2]]

Cayley graph is another and widely studied graph, associated with finite groups. Let G be a group and C be a subset of $G \setminus \{e\}$. Then the directed Cayley graph $\overrightarrow{X(G, C)}$ is defined to be a directed graph with vertex set G and arc set $\{(g, h) : g^{-1}h \in C\}$. If in addition, C is an inverse closed subset, then the undirected Cayley graph $X(G, C)$ is defined to be the underlying undirected graph of $\overrightarrow{X(G, C)}$. We refer to [3] for more on Cayley Graph of a group.

In the excellent survey [4] on power graph of groups and semigroups, Abawajy et. al. proposed the following problem:

Describe all directed and undirected power graphs of groups and semigroups that can be represented as Cayley graphs and, respectively, undirected Cayley graphs of groups or semigroups.

Here we describe all groups whose power graphs can be represented as Cayley graphs of some groups, both directed and undirected.

2 Main result

An undirected graph Γ is called regular if degree of each vertex is the same and Γ is called vertex transitive if its automorphism group acts transitively on the vertex set of Γ . Every Cayley graph is vertex transitive and every vertex transitive undirected graph Γ is regular. If Γ is directed vertex transitive graph then the in-degree of all the vertices are equal and the same holds for out-degree. Thus we have:

Theorem 2.1. 1. *Let G be a finite group. Then $\mathcal{P}(G)$ is a Cayley graph If and only if G is a cyclic p -group.*

2. *There is no finite group G , whose directed power graph is a Cayley graph.*

Proof. 1. Suppose G is a cyclic p -group, then the undirected power graph $\mathcal{P}(G)$ is complete and hence is a Cayley graph.

Conversely, assume that $\mathcal{P}(G)$ is a Cayley graph of some group. then $\mathcal{P}(G)$ is vertex transitive and so regular. If the order of G is n and e is the identity element of G , then degree of e is $n - 1$ in $\mathcal{P}(G)$ and hence degree of v is $n - 1$ for every v in G . It follows that $\mathcal{P}(G)$ is complete. Therefore G is a cyclic p -group.

2. If possible, on the contrary, assume that $\vec{\mathcal{P}}(G)$ is a directed Cayley graph for some group G of order n . Then $\vec{\mathcal{P}}(G)$ is vertex transitive and hence in-degree of all the vertices are equal and the same holds for out-degree. Now in $\vec{\mathcal{P}}(G)$, out-degree of e is 0 and in-degree of e is $n - 1$. Hence it follows that the out-degree of each vertex is 0 and the in-degree of each vertex is $n - 1$, which is impossible. Therefore, there is no finite group G such that the directed power graph $\vec{\mathcal{P}}(G)$ is Cayley graph of some group. \square

References

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